Problem Set 2 (Due 1/22/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work.

P1. Compute the following limits and prove your claim by using only the $\varepsilon - \delta$ definition.

(a) $\lim\limits_{z \to i} \overline{z}$

(b) $\lim\limits_{z \to 1+i} z^2$.

Solution. (a) The claim is that $\lim\limits_{z \to i} \overline{z} = -i$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. If $0 < |z - i| < \delta$, then

$$|\bar{z} - (-i)| = \left|\overline{z - i}\right| = |z - i| < \delta = \varepsilon.$$

This proves the claim.

(b) The claim is that $\lim\limits_{z \to 1+i} z^2 = 2i$. Let $\epsilon > 0$. If $0 < |z - (1+i)| < 3 - 2\sqrt{2}$ (note: $3 > 2\sqrt{2}$), then

$$\begin{aligned}
|z + (1+i)| &= |z - (1+i) + 2(1+i)| \\
&\leq |z - (1+i)| + 2\,|1+i| \\
&= |z - (1+i)| + 2\sqrt{2} \\
&< 3 - 2\sqrt{2} + 2\sqrt{2} \\
&= 3.
\end{aligned}$$

So choose $\delta = \min\left\{\frac{\varepsilon}{3}, 3 - 2\sqrt{2}\right\}$. Suppose that $0 < |z - (1+i)| < \delta$. By definition of δ, this means that $0 < |z - (1+i)| < 3 - 2\sqrt{2}$ (so the preceding estimation can be used) **AND** $0 < |z - (1+i)| < \varepsilon/3$. Hence, we obtain

$$\begin{aligned}
\left|z^2 - 2i\right| &= |z - (1+i)|\,|z + (1+i)| \\
&< \frac{\varepsilon}{3} \cdot 3 \\
&= \varepsilon.
\end{aligned}$$

This proves the claim.

P2. Let $M(z) = \frac{z-3}{1-2z}$. Prove that $\lim\limits_{z \to \infty} M(z) = -1/2$ and $\lim\limits_{z \to 1/2} M(z) = \infty$.

Proof. These follow easily from the theorems on limits involving the point at infinity. Here's a proof of the first one using the $\epsilon - \delta$ definition.

To prove $\lim_{z\to\infty} M(z) = -1/2$, let $\varepsilon > 0$. Choose $\delta = \min\left\{2, \frac{4\varepsilon}{5+2\varepsilon}\right\}$. If $|z| > \frac{1}{\delta}$, then we see that

$$\begin{aligned}
\left|\frac{z-3}{1-2z} - \left(\frac{1}{2}\right)\right| &= \left|\frac{-5}{2-4z}\right| \\
&= \frac{5}{2}\cdot\frac{1}{|2z-1|} \\
&\le \frac{5}{2}\cdot\frac{1}{|2\,|z|-1|} && (|2z-1| \ge |2\,|z|-1|) \\
&< \frac{5}{2}\cdot\frac{1}{2\,|z|-1} && \left(|z| > \frac{1}{\delta} \ge \frac{1}{2}\right) \\
&< \frac{5}{2}\cdot\frac{1}{2\left(\frac{5+2\varepsilon}{4\varepsilon}\right)-1} && \left(|z| > \frac{1}{\delta} \ge \frac{5+2\varepsilon}{4\varepsilon}\right) \\
&= \frac{5}{\frac{5+2\varepsilon}{\varepsilon} - 2} \\
&= \frac{5}{\frac{5+2\varepsilon-2\varepsilon}{\varepsilon}} = \varepsilon.
\end{aligned}$$

This proves the claim. ☕

P3. Consider the complex valued function

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Show that f is not differentiable at the origin.[1]

Proof. By definition,

$$\begin{aligned}
f'(0) &= \lim_{\Delta z\to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} \\
&= \lim_{\Delta z\to 0} \frac{\frac{\overline{\Delta z}^2}{\Delta z} - 0}{\Delta z} \\
&= \lim_{\Delta z\to 0} \frac{\overline{\Delta z}^2}{\Delta z^2}.
\end{aligned}$$

Along the real axis, $\Delta z = \overline{\Delta z}$ so the limit is 1. But along the line $z = x + ix$, we get $\Delta z = \Delta x + i\Delta x$ so that

$$\frac{\overline{\Delta z}^2}{\Delta z^2} = \frac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)^2} = \frac{(1-i)^2}{(1+i)^2} = \frac{-2i}{2i} = -1.$$

Since limits are unique, the limit doesn't exist and hence f is not differentiable at 0. Recall that in the lecture, we showed that f satisfies the CR-equations at 0, so indeed this shows that the satisfaction of the CR-equations alone is not a sufficient condition for differentiability. ☕

[1] In the lecture, we showed that f satisfies the Cauchy-Riemann equations at the origin. Thus, this will show that the satisfaction of the Cauchy-Riemann equations is not sufficient for the differentiability of f.

P4. Let U be a domain and $f : U \to \mathbb{C}$ a complex valued function that is differentiable at every point in U. Consider the domain $U^* \overset{\text{def}}{=} \{z \in \mathbb{C} : \overline{z} \in U\}$ and the function $f^* : U^* \to \mathbb{C}$ defined by $f^*(z) = \overline{f(\overline{z})}$. Show that f^* is differentiable at every point in U^*.

Proof. Assume that $f : U \to \mathbb{C}$ is differentiable at every point in U (since U is open we could have just said that f is analytic on U). We need to show that $f^* : U^* \to \mathbb{C}$ is analytic on U^*, so let $w \in U^*$. Then $\overline{w} \in U$ by definition and by assumption $f'(\overline{w})$ exists. We will try to compute the derivative of f^* at w:

$$\begin{aligned}
\lim_{z\to w} \frac{f^*(z) - f^*(w)}{z - w} &= \lim_{z\to w} \frac{\overline{f(\overline{z})} - \overline{f(\overline{w})}}{z - w} \\
&= \lim_{z\to w} \overline{\frac{f(\overline{z}) - f(\overline{w})}{\overline{z} - \overline{w}}} && \text{(conjugation is additive/multiplicative)} \\
&= \lim_{\overline{z}\to \overline{w}} \overline{\frac{f(\overline{z}) - f(\overline{w})}{\overline{z} - \overline{w}}} && (z \to w \iff \overline{z} \to \overline{w}) \\
&= \overline{\lim_{\overline{z}\to \overline{w}} \frac{f(\overline{z}) - f(\overline{w})}{\overline{z} - \overline{w}}} && \text{(since } f'(\overline{w}) \text{ exists)} \\
&= \overline{f'(\overline{w})} && \text{(definition of } f'(\overline{w}))
\end{aligned}$$

Thus, we have proved that f^* is differentiable at w and $(f^*)'(w) = \overline{f'(\overline{w})}$. Hence, f^* is analytic on U^*. ☕

P5. For each function, determine all points at which the derivative exists. When the derivative exists, find its value. You must justify your claims.

(a) $f(z) = z + i\overline{z}$;

(b) $g(z) = (z + i\overline{z})^2$;

(c) $h(z) = z \operatorname{Im} z$.

Solution. (a) Writing $z = x+iy$ we see that $f(z) = x+iy+i(x-iy) = (x+y)+i(x+y)$ so that $u(x,y) = x + y = v(x,y)$. Hence, $u_y = 1$ while $-v_x = -1$. Thus, the Cauchy-Riemann equations are not satisfied. Hence, f is nowhere differentiable.

(b) Writing $z = x + iy$ we see that

$$g(z) = f(z)^2 = ((x + y) + i(x + y))^2 = (x + y)^2 + 2i(x + y)^2 - (x + y)^2 = 2i(x + y)^2.$$

Hence, $u(x,y) = 0$ and $v(x,y) = 2(x+y)^2$. Moreover, $u_x = 0 = u_y$ and $v_x = 4(x+y) = v_y$. The solutions to the Cauchy-Riemann equations are then the points $L = \{(x,y) \in \mathbb{R}^2 : x + y = 0\}$ (the line $y = -x$). Since the partial derivatives of u and v are defined everywhere and are continuous everywhere, we conclude that g is differentiable at each point of the form $z = x - ix$ and the value is

$$g'(z) = g'(x - ix) = u_x(x, -x) + iv_x(x, -x) = 0 + i4(x + (-x)) = 0.$$

At all other points $z \in \mathbb{C} \setminus L$, $g'(z)$ does not exist since the Cauchy-Riemann equations are not satisfied.

(c) Writing $z = x + iy$ we see that $h(z) = (x + iy)y = xy + iy^2$ so that $u(x, y) = xy$ and $v(x, y) = y^2$. Hence, $u_x = y$, $u_y = x$, $v_x = 0$, and $v_y = 2y$. The Cauchy-Riemann equations are then $y = 2y$ and $x = 0$. The only solution is $(0, 0)$. Since the partial derivatives of u and v exist and are continuous everywhere, we can conclude that h' exists at the points $z = 0$.

$$h'(z_0) = h'(0) = u_x(0, 0) + iv_x(0, 0) = 0.$$

At all other points $z \in \mathbb{C} \setminus \{0\}$, the Cauchy-Riemann equations are not satisfied so h' doesn't exist there.

P6. (Optional.)[2] By definition, a function $f : U \to \mathbb{C}$ is differentiable at $z_0 \in U$ if the limit $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. Unpacking the limit definition, we see that f is differentiable at z_0 if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \text{ implies } \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon.$$

By appealing only to the definition, show that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{z}$ is not differentiable anywhere by completing the following steps:

(a) Let $z_0 \in \mathbb{C}$ and assume that $f'(z_0)$ exists. Choose $\delta > 0$ according to the definition using $\varepsilon = \frac{1}{2}$.

(b) Consider $z = z_0 + \frac{\delta}{2}$ and conclude from (a) that $|1 - f'(z_0)| < \varepsilon$.

(c) Consider $z = z_0 + \frac{i\delta}{2}$ and conclude from (a) that $|1 + f'(z_0)| < \varepsilon$.

(d) Using the triangle inequality together with (b) and (c), obtain a contradiction.

Proof. The steps give an outline of a proof. Let $z_0 \in \mathbb{C}$ and suppose to the contrary that $f'(z_0)$ exists. According to the definition with $\varepsilon = \frac{1}{2}$, we can choose $\delta > 0$ such that

$$0 < |z - z_0| < \delta \text{ implies } \left|\frac{\overline{z} - \overline{z_0}}{z - z_0} - f'(z_0)\right| < \frac{1}{2}.$$

Considering $z = z_0 + \frac{\delta}{2}$, we see that

$$|z - z_0| = \left|z_0 + \frac{\delta}{2} - z_0\right| = \frac{\delta}{2}.$$

Hence, $0 < |z - z_0| < \delta$, so we obtain

$$\frac{1}{2} > \left|\frac{\overline{z} - \overline{z_0}}{z - z_0} - f'(z_0)\right| = \left|\frac{\overline{z_0} + \frac{\delta}{2} - \overline{z_0}}{z_0 + \frac{\delta}{2} - z_0} - f'(z_0)\right| = \left|1 - f'(z_0)\right|.$$

Similarly, considering $z = z_0 + i\frac{\delta}{2}$ we see that

$$|z - z_0| = \left|z_0 + i\frac{\delta}{2} - z_0\right| = \left|i\frac{\delta}{2}\right| = \frac{\delta}{2}$$

[2] This is worth 3 points extra credit, added to your score on this assignment.

so that $0 < |z - z_0| < \delta$ from which we obtain

$$\frac{1}{2} > \left| \frac{\overline{z} - \overline{z_0}}{z - z_0} - f'(z_0) \right| = \left| \frac{\overline{z_0} - i\frac{\delta}{2} - \overline{z_0}}{z_0 + i\frac{\delta}{2} - z_0} - f'(z_0) \right| = |-1 - f'(z_0)| = |1 + f'(z_0)|.$$

Hence, we see that

$$\begin{aligned} 2 = |2 - f'(z_0) + f'(z_0)| &= |1 - f'(z_0) + 1 + f'(z_0)| \\ &\overset{T.I.}{\leq} |1 - f'(z_0)| + |1 + f'(z_0)| \\ &< \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

But even a fool knows that $2 \not< 1$! Since $z_0 \in \mathbb{C}$ was arbitrary, we may conclude that $f(z) = \overline{z}$ is nowhere differentiable. ☕